

GLOBAL WELL-POSEDNESS FOR A SLIGHTLY SUPERCRITICAL SURFACE QUASI-GEOSTROPHIC EQUATION

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Dedicated to Peter Constantin on occasion of his 60th birthday

ABSTRACT. We use a nonlocal maximum principle to prove the global existence of smooth solutions for a slightly supercritical surface quasi-geostrophic equation. By this we mean that the velocity field u is obtained from the active scalar θ by a Fourier multiplier with symbol $i\zeta^\perp|\zeta|^{-1}m(|\zeta|)$, where m is a smooth increasing function that grows slower than $\log \log |\zeta|$ as $|\zeta| \rightarrow \infty$.

1. INTRODUCTION

The surface quasi-geostrophic equation (SQG) has recently been a focus of research efforts by many mathematicians. It is probably the simplest physically motivated evolution equation of fluid mechanics for which, in the supercritical regime, it is not known whether solutions stay regular or can blow up. The equation is given by

$$\begin{aligned}\partial_t \theta + (u \cdot \nabla) \theta + \Lambda^\alpha \theta &= 0, \quad \theta(\cdot, 0) = \theta_0 \\ u &= \nabla^\perp \Lambda^{-1} \theta\end{aligned}$$

on $(x, t) \in \mathbb{T}^2 \times [0, \infty)$, where $\Lambda = (-\Delta)^{1/2}$. The SQG equation appeared in the mathematical literature for the first time in [4], and since then has attracted significant attention, in part due to certain similarities with three dimensional Euler and Navier-Stokes equations. The equation has L^∞ maximum principle [12, 3], which makes the $\alpha = 1$ dissipation critical. It has been known since [12] that the equation has global smooth solutions (for appropriate initial data) when $\alpha > 1$. The global regularity in the critical case has been settled independently by Kiselev-Nazarov-Volberg [11] (in the periodic setting) and Caffarelli-Vasseur [1] (in the whole space as well as in the local setting). A third proof of the same result was provided recently in [10]. All these proofs are quite different. The method of [1] is inspired by DeGiorgi iterative estimates, while the approach of [10] uses appropriate set of test functions and estimates on their evolution. The method of [11], on the other hand, is based on a new technique which can be called a nonlocal maximum principle. The idea is to prove that the evolution (1.1) preserves a certain modulus of continuity ω of the solution. The control is strong enough to give a uniform bound on $\|\nabla \theta\|_{L^\infty}$ in the critical case, which is sufficient for global regularity.

In the supercritical case, the only results available so far (for large initial data) have been on conditional regularity and finite time regularization of solutions. For instance, it was shown by Constantin and Wu [5] that if the solution is $C^{1-\alpha}$, then it is smooth. Finite time regularization has been proved by Silvestre [13] for α sufficiently close to 1, and for the whole dissipation range

Date: November 10, 2011.

2000 Mathematics Subject Classification. 35Q35, 76U5.

Key words and phrases. Surface quasi-geostrophic equation, supercritical, global regularity, active scalar, modulus of continuity, nonlocal maximum principle.

$0 < \alpha < 1$ by Dabkowski [6] (with an alternative proof of the latter result given in [9]). The issue of global regularity in the case $\alpha \in (0, 1)$ remains an outstanding open problem.

Our goal here is to advance global regularity very slightly into the supercritical regime for the SQG equation. For technical reasons (and inspired by [2]), it is more convenient for us to introduce supercriticality in the velocity u rather than in the dissipation. Namely, let $m(\zeta) = m(|\zeta|)$ be a smooth, radial, non-decreasing function on \mathbb{R}^2 , such that $m(\zeta) \geq 1$ for all $\zeta \in \mathbb{R}^2$. We shall consider the active scalar equation,

$$\partial_t \theta + (u \cdot \nabla) \theta + \Lambda \theta = 0, \quad \theta(\cdot, 0) = \theta_0 \quad (1.1)$$

$$u = \nabla^\perp \Lambda^{-1} m(\Lambda) \theta \quad (1.2)$$

on $(x, t) \in \mathbb{T}^2 \times [0, \infty)$, where $m(\Lambda) \theta$ is defined by its Fourier transform $(m(\Lambda) \theta)^\wedge(\zeta) = m(\zeta) \hat{\theta}(\zeta)$. Note that $m \equiv 1$ gives us the usual critical SQG equation. We shall consider symbols $m(\zeta)$ which for all sufficiently large $|\zeta|$ satisfy the growth condition

$$\lim_{|\zeta| \rightarrow \infty} \frac{m(\zeta)}{\ln \ln |\zeta|} = 0. \quad (1.3)$$

In addition we require that

$$\lim_{|\zeta| \rightarrow \infty} \frac{|\zeta| m'(\zeta)}{m(\zeta)} = 0 \quad (1.4)$$

and that the symbol m is of Hörmander-Mikhlin type, i.e., there exists $C > 0$ such that

$$|\zeta|^k |\partial_\zeta^k m(\zeta)| \leq C m(\zeta) \quad (1.5)$$

holds for all $\zeta \neq 0$, and all $k \in \{0, \dots, d+2\}$. The main result of this paper is:

Theorem 1.1 (Slightly supercritical SQG). *Assume that $\theta_0 \in C^\infty(\mathbb{T}^2)$. If the symbol m satisfies (1.3)–(1.5), then there exists a unique global C^∞ smooth solution θ of (1.1)–(1.2).*

Remark 1.2. The condition (1.4) can be improved to require only $\lim_{|\zeta| \rightarrow \infty} |\zeta| m'(\zeta) / m(\zeta) < 1$, but is adapted here for the sake of simplicity.

The result we prove here is reminiscent of the slightly supercritical Navier-Stokes regularity result of Tao [15]. The challenge in the SQG case is that while regularity for critical Navier-Stokes is easy to prove by energy method, there is no similarly simple proof of regularity for the critical SQG. The criticality of the SQG equation is controlled by the L^∞ norm, and the order of differentiation is the same in the nonlinearity and dissipation term. This makes global regularity for large data surprising at the first look. All three proofs of global regularity for critical SQG are somewhat subtle and involved. Scaling plays a crucial role in all existing proofs. The main contribution of this paper is to show that one can advance, at least a little, beyond the critical scaling.

To prove Theorem 1.1, we rely on the original method of [11]. This method is based on constructing a modulus of continuity $\omega(\xi)$, Lipschitz at zero and growing at infinity, which is respected by the critical SQG evolution: if the initial data θ_0 obeys ω , so does the solution $\theta(x, t)$ for every $t > 0$. By scaling, in the critical regime any rescaled modulus $\omega_B(\xi) = \omega(B\xi)$ is also preserved by the evolution. This allows, given smooth initial data θ_0 , to find B such that θ_0 obeys ω_B and thus, due to preservation of ω_B , gain sufficient control of solution for all times. The unboundedness of ω is crucial for this argument; applying it with bounded ω would correspond to controlling only initial data of limited size. It appears that the maximal growth of ω one can afford in the critical SQG

case is a double logarithm, dictated by balance of nonlinear and dissipative term estimates. The idea of the proof of Theorem 1.1, and the key observation of this paper, is that it is possible to trade some of this growth in ω for a slightly rougher velocity u (or, likely, slightly weaker dissipation). In the process, one loses critical scaling, but the argument can be made to work by manufacturing a family of moduli ω_B preserved by the evolution which are no longer a single rescaled modulus.

We anticipate that the approach we develop here will have other applications. In particular, it can be applied to a slightly supercritical Burgers equation. In this case, one can expect to prove global regularity for a more singular equation, supercritical by almost a logarithmic multiplier. This is due to the existence of moduli with logarithmic growth conserved by the evolution. Consideration of the Burgers equation, as well as applications to modified SQG, and the case of supercritical dissipation is postponed to a subsequent publication [7].

2. PRELIMINARIES

The local and conditional regularity for the SQG-type equations is by now standard. In particular, we have

Proposition 2.1 (Local existence of smooth solution). *Given $\theta_0 \in H^s(\mathbb{T}^2)$, for some $s > 1$, there exists $T > 0$ and a solution $\theta(\cdot, t) \in C([0, T], H^s) \cap C^\infty((0, T] \times \mathbb{T}^2)$ of (1.1)–(1.2). Moreover, the solution may be continued as a smooth solution beyond T as long as $\|\nabla \theta\|_{L^1(0, T; L^\infty(\mathbb{T}^2))} < \infty$.*

The proof of a similar result with standard SQG velocity and critical or supercritical dissipation can be found, for example, in [8]. The addition of the multiplier m into u does not create any essential difficulties in the argument.

Definition 2.2 (Modulus of continuity). *We call a function $\omega: (0, \infty) \rightarrow (0, \infty)$ a modulus of continuity if ω is increasing, continuous, concave, piecewise C^2 with one sided derivatives, and it additionally satisfies $\omega'(0+) = \infty$ or $\omega''(0+) = -\infty$. We say that a smooth function f obeys the modulus of continuity ω if $|f(x) - f(y)| < \omega(|x - y|)$ for all $x \neq y$.*

We recall that if $f \in C^\infty(\mathbb{T}^2)$ obeys the modulus ω , then $\|\nabla f\|_{L^\infty} < \omega'(0)$. In addition, observe that a function $f \in C^\infty(\mathbb{T}^2)$ automatically obeys any modulus of continuity $\omega(\xi)$ that lies above the function $\min\{\xi\|\nabla f\|_{L^\infty}, 2\|f\|_{L^\infty}\}$.

We will construct a family of moduli of continuity ω_B that will be preserved by the evolution. To prove this nonlocal maximum principle, we will use the following outline. The proofs of Lemmas 2.3 and 2.5 below can be found in [11].

Lemma 2.3 (Breakthrough scenario). *Assume ω is a modulus of continuity such that $\omega(0+) = 0$ and $\omega''(0+) = -\infty$. Suppose that the initial data θ_0 obeys ω . If the solution $\theta(x, t)$ violates ω at some positive time, then there must exist $t_1 > 0$ and $x \neq y \in \mathbb{T}^2$ such that*

$$\theta(x, t_1) - \theta(y, t_1) = \omega(|x - y|),$$

and $\theta(x, t)$ obeys ω for every $0 \leq t < t_1$.

Let us consider the breakthrough scenario for a modulus ω . A simple computation shows that

$$\begin{aligned} \partial_t (\theta(x, t) - \theta(y, t))|_{t=t_1} &= u \cdot \nabla \theta(y, t_1) - u \cdot \nabla \theta(x, t_1) + \Lambda \theta(y, t_1) - \Lambda \theta(x, t_1) \\ &\leq |u(x, t_1) - u(y, t_1)| \omega'(\xi) + \Lambda \theta(y, t_1) - \Lambda \theta(x, t_1). \end{aligned} \quad (2.1)$$

If we can show that the expression in (2.1) must be strictly negative, we obtain a contradiction: ω cannot be broken, and hence it is preserved. To estimate (2.1) we need

Lemma 2.4 (Modulus of continuity for the drift velocity). *Assume that θ obeys the modulus of continuity ω , and that the drift velocity is given as $u = \nabla^\perp \Lambda^{-1} m(\Lambda) \theta$. Then u obeys the modulus of continuity Ω defined as*

$$\Omega(\xi) = A \left(\int_0^\xi \frac{\omega(\eta) m(\eta^{-1})}{\eta} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta) m(\eta^{-1})}{\eta^2} d\eta \right) \quad (2.2)$$

for some positive constant $A \geq 1$ that only depends on the function m .

The proof of Lemma 2.4 shall be given in the Appendix. For the dissipative terms, we have:

Lemma 2.5 (Dissipation control). *Assume we are in a breakthrough scenario as in Lemma 2.3. Then*

$$\begin{aligned} \Lambda \theta(y, t_1) - \Lambda \theta(x, t_1) \leq \mathcal{D}(\xi) \equiv & \frac{1}{\pi} \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta \\ & + \frac{1}{\pi} \int_{\xi/2}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^2} d\eta. \end{aligned} \quad (2.3)$$

Given the three Lemmas above and (2.1), in order to verify the preservation of ω for all time, it is sufficient to check that $\Omega(\xi) \omega'(\xi) + \mathcal{D}(\xi) < 0$ for every $\xi > 0$.

The conditions imposed on the symbol m have two consequences which shall be useful later:

Lemma 2.6 (Further properties of m). *Let m be smooth radial radially non-decreasing function which satisfies (1.4). Then it holds that*

$$\lim_{|\zeta| \rightarrow \infty} \frac{m(\zeta)(1 + \ln |\zeta|)}{|\zeta|} = 0 \quad (2.4)$$

and we have

$$\int_0^{|\zeta|} m(r^{-1}) dr \leq 2|\zeta| m(|\zeta|^{-1}) \quad (2.5)$$

for all $|\zeta|$ which are sufficiently small, depending on m .

Proof of Lemma 2.6. From (1.4) it follows that there exists $r_0 > 0$ such that for all $|\zeta| \geq r_0$ we have $2|\zeta| m'(\zeta) \leq m(\zeta)$. In order to prove (2.4), let $f(r) = r m(r)$. We have $f'(r) \leq 3m(r)/2 = (3/2)r^{-1} f(r)$, for all $r \geq r_0$, and hence $f(r) \leq f(r_0) r_0^{-3/2} r^{3/2} = m(r_0) r_0^{-1/2} r^{3/2}$. Therefore $r^{-1}(1 + \ln r) m(r) = r^{-2}(1 + \ln r) f(r) \leq (1 + \ln r) r^{-1/2} m(r_0) r_0^{-1/2} \rightarrow 0$ as $r \rightarrow \infty$. In fact, it is easy to see that $m(\zeta)/|\zeta|^a \rightarrow 0$ as $\zeta \rightarrow \infty$ for every $a > 0$, but we will not need this stronger bound in the proof.

To prove (2.5), we note that the function $r^{1/2} m(r^{-1})$ is non-decreasing on $r \leq r_0^{-1}$. Therefore $m(r^{-1}) \leq |\zeta|^{1/2} m(|\zeta|^{-1}) r^{-1/2}$, and (2.5) follows if $|\zeta| \leq r_0^{-1}$, by integrating in r . \square

3. PROOF OF MAIN THEOREM

The main difference between our argument here and [11] is that since (1.1)–(1.2) is beyond the critical scaling, one cannot use $\omega_B(\xi) = \omega(B\xi)$ to construct the needed family of moduli of continuity, from a fixed modulus ω .

3.1. A suitable family of moduli of continuity. We fix a sufficiently small positive constant κ , to be chosen later. For $B \geq 1$, we define $\delta = \delta(B)$ to be the unique solution of

$$B\delta m(\delta^{-1}) = \kappa. \quad (3.1)$$

To see that δ exists and is unique, let $g(\delta) = \delta m(\delta^{-1})$. Due to (2.4), we have that $g(\delta) \rightarrow 0$ as $\delta \rightarrow 0+$, and due to (1.4), we have that $g'(\delta) = m(\delta^{-1}) - \delta^{-1}m'(\delta^{-1}) \geq m(\delta^{-1})/2 > 0$, for all $\delta \leq r_0^{-1}$. So g is increasing (and continuous) at least until r_0^{-1} , and hence if κ is chosen such that $\kappa \leq g(r_0^{-1}) = r_0^{-1}m(r_0)$, since $B \geq 1$, the equation $g(\delta) = \kappa B^{-1}$ will have a unique solution. Note that $\delta(B) \rightarrow 0$ as $B \rightarrow \infty$ since $g(0+) = 0$, and $\delta(B)$ is a strictly decreasing function of B .

Having defined $\delta(B)$ for each $B \geq 1$, we shall consider the modulus of continuity ω_B defined as the continuous function with $\omega_B(0) = 0$ and

$$\omega'_B(\xi) = B - \frac{B^2}{8\kappa} \xi m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi} \right), \text{ for all } 0 < \xi \leq \delta(B) \quad (3.2)$$

$$\omega'_B(\xi) = \frac{\gamma}{\xi(4 + \ln(\xi/\delta(B)))m(\delta(B)^{-1})}, \text{ for all } \xi > \delta(B) \quad (3.3)$$

where $\kappa = \kappa(A, m)$ and $\gamma = \gamma(\kappa, A, m)$ are sufficiently small constants to be chosen later. To verify that ω_B is a modulus of continuity, we need to check monotonicity, concavity, that $0 < \omega'_B(0+) < \infty$, and that $\omega''_B(0+) = -\infty$.

From (2.4) we know that $\xi m(\xi^{-1})(1 + \ln |\xi|^{-1}) \rightarrow 0$ as $\xi \rightarrow 0+$, and therefore, for every $B \geq 1$ we have that $\omega'_B(0+) = B$. Note that in fact we have $\omega'_B(\xi) < B$, and hence $\omega_B(\xi) \leq B\xi$ for all $0 < \xi \leq \delta(B)$. Taking the derivative of (3.2) we obtain

$$\begin{aligned} \omega''_B(\xi) &= -\frac{B^2}{8\kappa} \left((m(\xi^{-1}) - \xi^{-1}m'(\xi^{-1})) \left(4 + \ln \frac{\delta(B)}{\xi} \right) - m(\xi^{-1}) \right) \\ &\leq -\frac{B^2}{8\kappa} \left(\frac{1}{2} m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi} \right) - m(\xi^{-1}) \right) \\ &\leq -\frac{B^2}{32\kappa} m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi} \right) \end{aligned} \quad (3.4)$$

which implies that $\omega''_B(\xi) \rightarrow -\infty$ as $\xi \rightarrow 0+$ since $m(\xi^{-1}) \geq 1$ for all $\xi > 0$. Note that in the first inequality of (3.4) we have used $2\xi^{-1}m'(\xi^{-1}) \leq m(\xi^{-1})$, which holds for all $\xi \leq \delta(B)$ as long as $\delta(B)$ is sufficiently small (how small it needs to be depends only on m). One can always ensure that $\delta(B)$ is small enough since $\delta(B) \leq \delta(1)$ for all $B \geq 1$, and $\delta(1)$ can be made arbitrarily small by choosing κ to be sufficiently small, depending only on m .

From (3.3) and (3.4) it is clear that the concavity of ω_B may only fail at $\xi = \delta(B)$. However, from (3.1) and (3.2) we obtain

$$\omega'_B(\delta(B)-) = B - \frac{B^2}{2\kappa} \delta(B) m(\delta(B)^{-1}) = \frac{B}{2}. \quad (3.5)$$

On the other hand by (3.3) we have

$$\omega'_B(\delta(B)+) = \frac{\gamma}{4\delta(B)m(\delta(B)^{-1})} = \frac{\gamma B}{4\kappa} < \frac{B}{4} \quad (3.6)$$

for all $\gamma < \kappa$. Together, (3.5) and (3.6) show that ω_B is concave.

To prove that ω_B is monotonically increasing, it is sufficient to verify that $\omega'_B(\xi) > 0$ for all $0 < \xi < \delta(B)$. But $\omega'_B(0) = B \geq 1$ and $\omega'_B(\xi)$ is decreasing on $(0, \delta(B))$ due to (3.4), so that we only need to verify that $\omega'_B(\delta(B)-) > 0$. This follows directly from (3.5).

Let us denote $\Omega_B(\xi)$ and $\mathcal{D}_B(\xi)$ respectively the modulus of the velocity u given by (2.2) and dissipation estimate (2.3) corresponding to $\omega_B(\xi)$. It is sufficient to prove two things: that each initial data θ_0 obeys some modulus of continuity ω_B for a suitable $B \geq 1$, and that the expression in (2.1) when computed for each ω_B is strictly negative for all $\xi > 0$.

3.2. Modulus of continuity for the initial data. First we show that any initial data $\theta_0 \in C^\infty(\mathbb{T}^2)$ obeys a modulus of continuity ω_B for some sufficiently large B . As noted earlier, this is achieved if we find a sufficiently large B such that $\omega_B(\xi) > \min\{\xi \|\nabla \theta_0\|_{L^\infty}, 2\|\theta_0\|_{L^\infty}\}$ for all $\xi > 0$. Observe that due to concavity of ω_B it is sufficient to find B such that

$$\omega_B\left(\frac{2\|\theta_0\|_{L^\infty}}{\|\nabla \theta_0\|_{L^\infty}}\right) \geq 2\|\theta_0\|_{L^\infty}.$$

However, note that for every fixed $a > 0$, we have $a > \delta(B)$ if B is sufficiently large, and

$$\int_{\delta(B)}^a \frac{\gamma}{\xi(4 + \ln(\xi/\delta(B)))m(\delta(B)^{-1})} d\xi = \frac{\gamma}{m(\delta(B)^{-1})} \ln(1 + \ln(a/\delta(B))) \rightarrow \infty$$

as $B \rightarrow \infty$ due to our assumption (1.3) on growth of m , and since $\delta(B) \rightarrow 0$ as $B \rightarrow \infty$. This shows that any smooth θ_0 obeys a modulus of continuity ω_B if B is chosen large enough.

3.3. Conservation of the modulus of continuity. We shall now prove that if κ is chosen sufficiently small (depending only on m , and A), and γ is chosen sufficiently small (depending only on κ , m , and A), then the expression (2.1) is strictly negative, i.e. $\Omega_B(\xi)\omega'_B(\xi) + \mathcal{D}_B(\xi) < 0$, for all $\xi > 0$. Note that neither κ , nor γ will depend on $B \geq 1$.

The case $0 < \xi \leq \delta(B)$. We first observe that $\omega'_B(\xi) \leq B$ for all $\xi \in (0, \delta(B)]$. Using concavity of ω and the mean value theorem we may estimate

$$\mathcal{D}_B(\xi) \leq \frac{1}{\pi} \xi \omega''_B(\xi).$$

In addition, it follows from the bound (3.4) on $\omega''_B(\xi)$ and the above estimate that

$$\mathcal{D}_B(\xi) \leq -\frac{1}{32\pi\kappa} B^2 \xi m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi}\right). \quad (3.7)$$

The main issue is to estimate the contribution from $\Omega_B(\xi)$. From (2.2) and (3.2) we have that

$$\begin{aligned} \Omega_B(\xi)\omega'_B(\xi) &\leq AB \left(B \int_0^\xi m(\eta^{-1}) d\eta + B\xi \int_\xi^{\delta(B)} \frac{m(\eta^{-1})}{\eta} d\eta + \xi \int_{\delta(B)}^\infty \frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2} d\eta \right) \\ &\leq AB \left(2B\xi m(\xi^{-1}) + B\xi m(\xi^{-1}) \ln \frac{\delta(B)}{\xi} + \xi m(\xi^{-1}) \int_{\delta(B)}^\infty \frac{\omega_B(\eta)}{\eta^2} d\eta \right). \end{aligned} \quad (3.8)$$

In the second inequality of (3.8) we have used the monotonicity of m and the inequality (2.5), which holds for all $\xi \leq \delta(B)$, whenever $\delta(B)$ is sufficiently small, depending only on m . But note that letting κ be sufficiently small, depending on m and not on B , we ensure that $\delta(1)$ is sufficiently small, and the bound $\delta(B) \leq \delta(1)$ for all $B \geq 1$, justifies the applicability of (2.5) for all $B \geq 1$.

In order to estimate $\int_{\delta(B)}^{\infty} \omega_B(\eta)/\eta^2 d\eta$, we integrate by parts and use the slow growth of ω_B (cf. (1.3)) to obtain

$$\begin{aligned} \int_{\delta(B)}^{\infty} \frac{\omega_B(\eta)}{\eta^2} d\eta &\leq \frac{\omega_B(\delta(B))}{\delta(B)} + \int_{\delta(B)}^{\infty} \frac{\gamma}{\eta^2(4 + \ln(\eta/\delta(B)))m(\delta(B)^{-1})} d\eta \\ &\leq B + \frac{\gamma}{4\delta(B)m(\delta(B)^{-1})} = B + \frac{\gamma B}{4\kappa} \leq 2B \end{aligned} \quad (3.9)$$

if $\gamma < \kappa$, since $m(\delta(B)^{-1}) \geq 1$. Combining (3.7) with (3.8) and (3.9) we obtain

$$\Omega_B(\xi)\omega'_B(\xi) + \mathcal{D}_B(\xi) \leq \left(A - \frac{1}{32\pi\kappa}\right) B^2 \xi m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi}\right) < 0 \quad (3.10)$$

for all $\xi \in (0, \delta(B)]$ if we choose κ so that $32\pi\kappa A < 1$. To avoid a circular argument, note that κ was chosen independently of γ and B , it only depends on m and A .

The case $\xi > \delta(B)$. We observe that for each $B \geq 1$ the modulus of continuity ω_B satisfies

$$\omega_B(2\xi) \leq \frac{3}{2}\omega_B(\xi), \text{ for all } \xi \geq \delta(B). \quad (3.11)$$

Indeed due to the definition (3.3) of ω_B , we have

$$\omega_B(2\xi) \leq \omega_B(\xi) + \frac{\gamma}{4m(\delta(B)^{-1})}$$

for every $\xi \geq \delta(B)$. But from the monotonicity of ω_B and the mean value theorem we have $\omega_B(\xi) \geq \omega_B(\delta(B)) \geq \delta(B)\omega'_B(\delta(B)-)$, since ω'_B is strictly decreasing on $(0, \delta(B))$. By (3.1) and (3.5) it follows that taking $\gamma < \kappa$ is sufficient for (3.11) to hold. Using (3.11), we estimate

$$\mathcal{D}_B(\xi) \leq \frac{1}{\pi} \int_{\xi/2}^{\infty} \frac{\omega_B(2\eta + \xi) - \omega_B(2\eta - \xi) - \omega_B(2\xi) - \frac{1}{2}\omega_B(\xi)}{\eta^2} d\eta \leq -\frac{1}{2\pi} \frac{\omega_B(\xi)}{\xi} \quad (3.12)$$

which holds for all $\xi > \delta(B)$. Next, let us bound the term arising from $\Omega_B(\xi)\omega'_B(\xi)$ in (2.1), namely

$$A\omega'_B(\xi) \left(\int_0^{\xi} \frac{\omega_B(\eta)m(\eta^{-1})}{\eta} d\eta + \xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2} d\eta \right). \quad (3.13)$$

We first estimate

$$\begin{aligned} \int_0^{\xi} \frac{\omega_B(\eta)m(\eta^{-1})}{\eta} d\eta &\leq B \int_0^{\delta(B)} m(\eta^{-1}) d\eta + \int_{\delta(B)}^{\xi} \frac{\omega_B(\eta)m(\eta^{-1})}{\eta} d\eta \\ &\leq 2B\delta(B)m(\delta(B)^{-1}) + \omega_B(\xi)m(\delta(B)^{-1}) \ln \frac{\xi}{\delta(B)} \\ &= 2\kappa + \omega_B(\xi)m(\delta(B)^{-1}) \ln \frac{\xi}{\delta(B)} \end{aligned} \quad (3.14)$$

for all $\xi > \delta(B)$. Above we used (3.1) and (2.5), which may be applied since $\delta(B)$ is sufficiently small with respect to m for any $B \geq 1$. Furthermore, upon integration by parts we have

$$\begin{aligned} \xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2} d\eta &\leq \xi m(\xi^{-1}) \int_{\xi}^{\infty} \frac{\omega_B(\eta)}{\eta^2} d\eta \\ &= \xi m(\delta(B)^{-1}) \left(\frac{\omega_B(\xi)}{\xi} + \frac{\gamma}{m(\delta(B)^{-1})} \int_{\xi}^{\infty} \frac{1}{\eta^2(4 + \ln(\eta/\delta(B)))} d\eta \right) \\ &\leq \omega_B(\xi)m(\delta(B)^{-1}) + \gamma. \end{aligned} \quad (3.15)$$

Therefore, inserting the bounds (3.14) and (3.15) into (3.13), and letting $\gamma \leq \kappa$, we obtain

$$\begin{aligned} \Omega_B(\xi)\omega'_B(\xi) &\leq A\omega'_B(\xi) \left(\gamma + 2\kappa + \omega_B(\xi)m(\delta(B)^{-1}) \left(1 + \ln \frac{\xi}{\delta(B)} \right) \right) \\ &\leq \frac{A\gamma}{\xi(4 + \ln \xi/\delta(B))m(\delta(B)^{-1})} \left(3\kappa + \omega_B(\xi)m(\delta(B)^{-1}) \left(1 + \ln \frac{\xi}{\delta(B)} \right) \right) \\ &\leq \frac{2A\gamma\omega_B(\xi)}{\xi} \end{aligned} \quad (3.16)$$

since $\kappa \leq 2\omega_B(\delta(B))m(\delta(B)^{-1}) \leq 2\omega_B(\xi)m(\delta(B)^{-1})$. Indeed, the latter holds since as above we have

$$\omega_B(\delta(B)) \geq \delta(B)m'(\delta(B)-) = \frac{B\delta(B)}{2} = \frac{\kappa}{2m(\delta(B)^{-1})}.$$

Lastly, from (3.12) and (3.16) it follows that

$$\Omega_B(\xi)\omega'_B(\xi) + \mathcal{D}_B(\xi) < \left(2A\gamma - \frac{1}{2\pi} \right) \frac{\omega_B(\xi)}{\xi} < 0 \quad (3.17)$$

as long as γ is chosen small enough so that $4\pi A\gamma < 1$.

4. APPENDIX

Here we give details regarding the proof of Lemma 2.4. Let $m(\zeta)$ be a continuous, radial, non-decreasing function on \mathbb{R}^d , smooth on \mathbb{R}^d , with $m(\zeta) = m(|\zeta|) \geq 1$ for all $\zeta \in \mathbb{R}^d$. Assume that $m(\zeta)$ satisfies the Hörmander-Mikhlin-type condition (cf. [14])

$$|\zeta|^k |\partial_{\zeta}^k m(\zeta)| \leq C m(\zeta) \quad (4.1)$$

for some $C \geq 1$, all $\zeta \neq 0$, and all $k \in \{0, \dots, d+2\}$. In addition we require that

$$\lim_{|\zeta| \rightarrow \infty} \frac{|\zeta| m'(\zeta)}{m(\zeta)} = 0. \quad (4.2)$$

The following lemma gives estimates on the distribution K whose Fourier transform is $i\zeta_j |\zeta|^{-1} m(\zeta)$, for any $j \in \{1, \dots, d\}$.

Lemma 4.1 (Kernel estimate). *Let $K(x)$ be the kernel of the operator $\partial_j \Lambda^{-1} m(\Lambda)$, where m is smooth on \mathbb{R}^d , radial, non-decreasing in radial variable, and satisfies the conditions (4.1)–(4.2). Then we have*

$$|K(x)| \leq C |x|^{-d} m(|x|^{-1}) \quad (4.3)$$

and

$$|\nabla K(x)| \leq C |x|^{-d-1} m(|x|^{-1}) \quad (4.4)$$

for all $x \neq 0 \in \mathbb{R}^d$.

Proof of Lemma 4.1. Consider a smooth non-increasing radial cutoff function $\eta(\zeta) = \eta(|\zeta|)$ which is identically 1 on $|\zeta| \leq 1/2$, and vanishes identically on $|\zeta| \geq 1$. For $R > 0$, let $\eta_R(|\zeta|) = \eta(|\zeta|/R)$. Then, for $R > 0$ to be chosen later, we decompose

$$K(x) = C \int_{\mathbb{R}^d} \eta_R(\zeta) m(\zeta) i\zeta_j |\zeta|^{-1} e^{i\zeta \cdot x} d\zeta + C \int_{\mathbb{R}^d} (1 - \eta_R(\zeta)) m(\zeta) i\zeta_j |\zeta|^{-1} e^{i\zeta \cdot x} d\zeta = K_1(x) + K_2(x).$$

Since $m(\zeta)$ is increasing, and η_R is supported on B_R , we may bound $|K_1(x)| \leq CR^d m(R)$. On the other hand, upon integrating by parts $d+2$ times, using (4.1) and the fact that $\partial_\zeta(1 - \eta_R(\zeta))$ is supported on $R/2 \leq |\zeta| \leq R$, we obtain

$$\begin{aligned} |K_2(x)| &\leq C|x|^{-d-2} \int_{\mathbb{R}^d} |\partial_\zeta^{d+2} ((1 - \eta_R)(\zeta) m(\zeta) i\zeta_j |\zeta|^{-1})| d\zeta \\ &\leq C|x|^{-d-2} \left(R^{-d-2} \int_{R/2 \leq |\zeta| \leq R} m(\zeta) d\zeta + \int_{|\zeta| \geq R/2} |\zeta|^{-d-2} m(\zeta) d\zeta \right). \end{aligned} \quad (4.5)$$

Observe that condition (4.2) shows there exists some $r > 0$ such that for all $|\zeta| \geq r$ we have $2|\zeta|m'(\zeta) \leq m(\zeta)$, and hence the function $|\zeta|^{-1/2}m(|\zeta|)$ is non-increasing for $|\zeta| \geq r$. Consider first small x , $|x| \leq 1/2r$. Letting $R = |x|^{-1}$, we have that $R/2 \geq r$. Using the facts that $m(|\zeta|)$ is non-decreasing, and $|\zeta|^{-1/2}m(|\zeta|)$ is non-increasing on $|\zeta| \geq r$, we obtain

$$|K_2(x)| \leq C|x|^{-d}m(|x|^{-1}) \quad (4.6)$$

which upon recalling the earlier bound on K_1 concludes the proof of (4.3) for small x . For $|x| \geq 1/2r$, we can set $R = 1$ and obtain that

$$|K_2(x)| \leq C|x|^{-d-2},$$

since due to (4.2) and the continuity of m we have $|m(\zeta)| \leq C(m)|\zeta|^{1/2}$. On the other hand,

$$K_1(x) = C \int_{\mathbb{R}^d} (c_0 i\zeta_j |\zeta|^{-1} + \varphi(\zeta)) e^{i\zeta \cdot x} d\zeta,$$

where c_0 is a constant and $\varphi(\zeta) \in C_0^\infty$. This gives the bound

$$|K(x)| \leq C|x|^{-d},$$

which together with (4.6) implies (4.3) for $|x| \geq 1/2r$. The bounds for $\nabla K(x)$ are obtained in the same fashion, the only difference being an extra factor of ζ in the estimates. \square

Having estimated the kernel of the operator $\theta \mapsto u$, we are now ready to estimate the modulus of continuity of the velocity u , in terms of the modulus of continuity of the active scalar θ .

Proof of Lemma 2.4. The proof is similar to that of [11, Lemma]. Fix $x \neq y$, and let $\xi = |x - y|$. Since $u = \nabla^\perp (\Lambda^{-1}m(\Lambda)\theta)$ we have that $\int_{|x|=1} K(x) d\sigma(x) = 0$, and hence we may bound

$$\begin{aligned} u(x) - u(y) &= \int_{|x-z| \leq 2\xi} K(x-z)(\theta(z) - \theta(x)) dz - \int_{|y-z| \leq 2\xi} K(y-z)(\theta(z) - \theta(y)) dz \\ &\quad + \int_{|x-z| \geq 2\xi} K(x-z)(\theta(z) - \theta(\bar{z})) dz - \int_{|y-z| \geq 2\xi} K(y-z)(\theta(z) - \theta(\bar{z})) dz \end{aligned}$$

where the integrals are taken in the principal value sense, and $\bar{z} = (x + y)/2$. Using the estimates on the kernel K from Lemma 4.1, we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq C \int_0^{2\xi} \frac{m(\eta^{-1})\omega(\eta)}{\eta} d\eta + \int_{|\bar{z}-z| \geq 3\xi} |K(x-z) - K(y-z)| |\theta(z) - \theta(\bar{z})| dz \\ &\quad + \int_{3\xi/2 \leq |\bar{z}-z| \leq 3\xi} (|K(x-z)| + |K(y-z)|) |\theta(z) - \theta(\bar{z})| dz. \end{aligned} \quad (4.7)$$

To estimate the second integral on the right hand side, note that for $|z - \bar{z}| \geq 3\xi$, by the mean value theorem and (4.4), we have

$$|K(x-z) - K(y-z)| \leq C\xi |z - \bar{z}|^{-3} m(|z - \bar{z}|^{-1}).$$

Here we use that $m(sr) \leq s^C m(r)$ holds by (4.1) for $s > 1$. The third integral on the right hand side of (4.7) is bounded using (4.3) and we obtain

$$|u(x) - u(y)| \leq C \int_0^{3\xi} \frac{m(\eta^{-1})\omega(\eta)}{\eta} d\eta + C\xi \int_{3\xi}^{\infty} \frac{m(\eta^{-1})\omega(\eta)}{\eta^2} d\eta \quad (4.8)$$

for all $\xi \neq 0$. The final result then follows from (4.8) using the concavity of ω and the monotonicity of m . \square

Acknowledgement. MD was supported in part by the NSF grant DMS-0800243. AK acknowledges support of the NSF grant DMS-1104415, and thanks Terry Tao for a useful discussion.

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